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Hierarchy of fractional quantum Hall states in multi-layer electron systems

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Abstract. It is shown that the recursion relations for the filling fractions of fractional quantum Hall states in double-layer electron systems proposed by M Greiter and I A McDonald can be expressed in a 'matrix continued fraction form'. We extend this matrix continued fraction form to multi-layer electron systems and show that under some physical restrictions the matrix continued fraction form is reduced to the usual numerical continued fraction.

1. Introduction

Recently, the hierarchy of fractionally quantized Hall states of Haldane [1] and Halperin [2] has been extended to the double-layer electron systems by Greiter and McDonald [3]. The crucial thing about the double-layer systems is that one can have two kinds of electron by introducing an index to indicate in which of the two layers the electrons reside. The layer index can be treated as a kind of quantum number, called 'pseudo-spin', in close analogy to the spin of the electrons. Thus, the double-layer electron systems have an additional degree of freedom, which allows the Laughlin wavefunction to abide in the Pauli principle yet describes a fractional quantum Hall effect (FQHE) state with even-denominator filling fraction. Similar to the single-layer Haldane–Halperin (HH) hierarchy [1, 2], the hierarchy for double-layer systems (hereafter we term it the Greiter–McDonald (GM) hierarchy) is also based on the idea that quasiparticles themselves condense into a state similar to the primary state. This procedure can be iterated, and filling fractions with odd as well as even denominator appear within this picture at various levels of the GM hierarchy.

In this paper, we will point out that the recursion relations for the filling fractions of the GM hierarchy can be expressed in a matrix continued fraction form which is applicable to multi-layer electron systems. We show that under some physical restrictions the filling fractions of the matrix continued fraction form are reduced to the usual numerical continued fractions.

The remainder of this paper is presented as follows. In the next section we review the main arguments leading to the GM hierarchy and point out that the recursion relations for the filling fractions of the GM hierarchy can be expressed in the matrix continued fraction form. Then we show that under some physical restrictions the filling fractions of the matrix continued fraction form are reduced to the usual numerical continued fractions. We generalize the arguments for double-layer systems to triple- and multi-layer systems in section 3, emphasizing that the matrix continued fraction form for the filling fractions is applicable to the many-layer case. We conclude with remarks on some important properties of the numerical continued fractions appearing in the text.

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2. Review of hierarchy for double-layer systems

In a seminal paper Greiter and McDonald [3] have extended the HH hierarchy to the case of double-layer electron systems. Even though the underlying ideas were developed for double-layer systems, they are sufficiently general that they can be applied *mutatis mutandis* to triple-layer as well as multi-layer systems. Let us therefore review the main arguments of Greiter and McDonald.

Let the two-component Laughlin wavefunction in the spherical geometry be

$$\psi^{(m_1,m_2,n)}[u_i^1, v_i^1, u_i^2, v_i^2] = \prod_{i<j}^{N_1}(u_i^1 v_j^1 - u_j^1 v_i^1)^{m_1} \prod_{i<j}^{N_2}(u_i^2 v_j^2 - u_j^2 v_i)^{m_2} \prod_{i,j}^{N_1 N_2}(u_i^1 v_j^2 - u_j^2 v_i^1)^n \quad (1)$$

where u_i^1, v_i^1 and u_i^2, v_i^2 are the spinor coordinates for the electrons in the first and the second layer, respectively. Each electron coordinate is represented in each term of the wavefunction by a polynomial in u_i^k and v_i^k of total degree $2S_k$, where $k = 1$ and 2 for the first and the second layer, respectively, and $2S_k$ is the flux through the surface k in units of the flux quantum. This is formally expressed through the operator

$$2\hat{S}_k = u_i^k(\partial/\partial u_i^k) + v_i^k(\partial/\partial v_i^k) \qquad k = 1, 2 \quad (2)$$

where the eigenvalue is constrained to be $2S_k$ for every electron i. Using the operator (2) with the wavefunction (1) for the first-layer coordinates u_i^1, v_i^1, we find

$$2S_1 = N_\phi = m_1(N_1 - 1) + nN_2 \quad (3)$$

where N_ϕ denotes the flux number and N_1 and N_2 are the numbers of electrons in each layer (1 and 2 refer to the first and second layers, respectively). For the second layer, we find

$$2S_2 = N_\phi = m_2(N_2 - 1) + nN_1. \quad (4)$$

Equations (3) and (4) can be expressed in the following matrix form:

$$\begin{pmatrix} 2S_1 \\ 2S_2 \end{pmatrix} = \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} - \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}. \quad (5)$$

In the spherical geometry, we can identify the excitation operators analogous to those proposed by Laughlin. We imagine adiabatically piercing the system with a unit of flux quantum, either forming a charge deficiency (quasihole) or a charge surplus (quasiparticles), depending on the direction of the extra flux. This has the effect of multiplying (1) by

$$\prod_{i=1}^{N_{ex,1}}(\beta_1 u_i^1 - \alpha_1 v_i^1) \prod_{i=1}^{N_{ex,2}}(\beta_2 u_i^2 - \alpha_2 v_i^2) \quad (6)$$

for the quasiholes or

$$\prod_{i=1}^{N_{ex,1}}[\beta_1^*(\partial/\partial u_i^1) - \alpha_1^*(\partial/\partial v_i^1)] \prod_{i=1}^{N_{ex,2}}[\beta_2^*(\partial/\partial u_i^2) - \alpha_2^*(\partial/\partial v_i^2)] \quad (7)$$

for the quasielectrons, where $N_{ex,1}$ and $N_{ex,2}$ are the numbers of excitations in the respective layers. α and β are constants. The operations (6) and (7) increase and decrease the number of flux quanta $2S_k$ through the surface k by one unit, respectively, creasing fractionally charged defects. From equations (1), (6) and (7), we obtain the quasihole (QH) and quasielectron (QE) wavefunctions as follows:

$$\psi_{QH}^{(m_1 m_2 n)}[u_i^1, v_i^1, u_i^2, v_i^2] = \psi^{(m_1 m_2 n)}[u_i^1, v_i^1, u_i^2, v_i^2] \prod_{i=1}^{N_{ex,1}}(\beta_1 u_i^1 - \alpha_1 v_i^1) \prod_{i=1}^{N_{ex,2}}(\beta_2 u_i^2 - \alpha_2 v_i^2) \quad (8)$$

$$\psi_{QE}^{(m_1 m_2 n)}[u_i^1, v_i^1, u_i^2, v_i^2] = \psi^{(m_1 m_2 n)}[u_i^1, v_i^1, u_i^2, v_i^2] \prod_{i=1}^{N_{ex,1}} [\beta_1^*(\partial/\partial u_i^1) - \alpha_1^*(\partial/\partial v_i^1)]$$

$$\times \prod_{i=1}^{N_{ex,2}} [\beta_2^*(\partial/\partial u_i^2) - \alpha_2^*(\partial/\partial v_i^2)]. \tag{9}$$

Using the operator (2) with the wavefunctions (8) and (9), we find that the presence of the quasiparticles will alter the flux–number relationship to become

$$\begin{pmatrix} 2S_1 \\ 2S_2 \end{pmatrix} = \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} - \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} N_{ex,1} \\ N_{ex,2} \end{pmatrix} \tag{10}$$

where m_1 and m_2 are odd, n is an integer and $\alpha_{1,2} = 1$ for quasiholes and $\alpha_{1,2} = -1$ for quasielectrons. Symbolically, we can write (10) in the following form:

$$2S = [m] \cdot N - \{m\} + \alpha N_{ex} \tag{11}$$

where $[m]$ is a 2×2 matrix, and $\{m\}$ a two-dimensional vector.

We can extend the reasoning that led to (10) to find a flux–number relationship for the excitations given by

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} p_1 & q \\ q & p_2 \end{pmatrix} \begin{pmatrix} N_{ex,1} \\ N_{ex,2} \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{12}$$

where p_1 and p_2 are even and q is an integer. Similar to (11), (12) can be symbolically written as

$$N = [p] \cdot N_{ex} - \{p\}. \tag{13}$$

From (11) and (13), we have

$$2S = [m, \alpha p] \cdot N - \{m, \alpha p\} \tag{14}$$

where

$$[m, \alpha p] = [m] + \alpha [p]^{-1} \tag{15}$$

is the inverse filling fraction ν^{-1}, and

$$\{m, \alpha p\} = \{m\} - \alpha [p]^{-1} \cdot \{p\} \tag{16}$$

is the flux shift N_{shift}. According to (15), the filling fraction can be formally expressed in the following matrix continued fraction form:

$$\nu = \frac{1}{[m, \alpha p]} = \frac{1}{[m] + \alpha/[p]} \tag{17}$$

where the inversion of matrices $[p]$ and $[m, \alpha p]$ is formally expressed by $1/[p]$ and $1/[m, \alpha p]$, respectively, and $\alpha = 1$ for quasiholes and -1 for quasielectrons.

At this point, we note that, following the steps that led to (17), the filling fraction at the n-level of the hierarchy can be written as

$$\nu = \frac{1}{[m, \alpha_1 p_1, \ldots, \alpha_n p_n]} = \frac{1}{[m] + \alpha_1/[[p_1] + \alpha_2/([p_2] + \cdots + \alpha_n/[p_n])]} \tag{18}$$

where

$$[m] = \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix} \qquad [p_i] = \begin{pmatrix} p_{i1} & q_i \\ q_i & p_{i2} \end{pmatrix} \qquad \alpha_i = \begin{pmatrix} \alpha_{i1} & 0 \\ 0 & \alpha_{i2} \end{pmatrix} \qquad i = 1, 2, \ldots, n.$$

When $n = 1$, (18) is reduced to (17). (18) can be easily proved by mathematical induction. It should be noted that (18) has two important special cases.

(i) For $[m] = m$, $[p_i] = p_i$ and $\alpha_{i1} = \alpha_{i2} = \alpha_i$, where m is odd, p_i even and $\alpha_i = \pm 1$, (18) is reduced to the HH hierarchy. Since we have ignored the tunnelling between the two layers, in the vanishing-Zeeman-splitting (VZS) limit, each layer is expected to have the same spin states. For instance, the spin states of the Laughlin sequence with $\nu = 1/(2k+1)$ (k an integer) are fully polarized (spinless); the states with $\nu = 2/(2k+1)$ are spin-unpolarized (spin-singlet). Also, the state at $\nu = 3/5$ has been found to be partially polarized by an exact diagonalization study [4], in agreement with experiments [5]. It is well known that in the HH hierarchial picture, the elementary excitations in the state with filling fraction $\nu = k/(2k+1)$ have charge $\pm e/(2k+1)$ [1, 2].

(ii) For

$$[m] = \begin{pmatrix} m & n \\ n & m \end{pmatrix} \qquad [p_i] = \begin{pmatrix} p_i & q_i \\ q_i & p_i \end{pmatrix} \qquad \alpha_i = \pm 1 \qquad i = 1, 2, \ldots, n \tag{19}$$

(18) can be expressed as

$$\nu = \frac{1}{m + n + \alpha_1/\{p_1 + q_1 + \alpha_2/[p_2 + q_2 + \cdots + \alpha_n/(p_n + q_n)]\}}. \tag{20}$$

Physically, (19) implies the following three things:

(i) for a primary state (m, m, n), the intralayer correlations in both layers are equal to m and the interlayer correlations are n;

(ii) all daughter states (p_i, p_i, q_i) have similar correlations as the parent state (m, m, n), namely, at the ith level of the hierarchy the intralayer correlations in both layers are equal to p_i and the interlayer correlations are q_i;

(iii) $\alpha_i = 1$ or -1 denotes that the constituents of the ith-level daughter state are all charged according to the same sign, implying repulsive correlations between quasiparticles in both interlayer and intralayer.

Under such three physical restrictions, the n-level matrix continued fraction form (18) is reduced to the usual numerical continued fractions (20). In this case, both layers have the same filling fractions. Most of the filling fractions calculated in [3] are contained in (20). Especially, the primary state (m, m, n) of (20) for $m = 3$ and $n = 1$, which corresponds to the total filling fraction $\nu = 1/2$, has been observed by Eisenstein *et al* [6] and others [7]. At present, although the hierarchial states predicted by (20) have not been experimentally observed, we do believe that these states are existing and reasonable. The reasons are the following.

First, the three physical restrictions leading to (20) are reasonable and the resultant numerical continued fractions are similar to the Haldane continued fractions for a single layer system.

Second, the HH hierarchy played a very important role in the infancy of the fractional quantum Hall theory. From the current point of view, the HH hierarchy is not completely satisfactory, but the hierarchy approach is still a good starting point for refined theories [8]. In our opinion, such situations will appear again in the studies of two-layered systems, namely, at the primary stage, we do need a hierarchy approach for two-layered systems similar to the HH hierarchy for a single-layer system.

Third, this hierarchy approach can give the scope of the fractional states for two-layered systems and draw the experimentalists' attentions to these states. We believe that, with improvement of the sample quality, the hierarchial states predicted by (20) can be experimentally observed.

According to our experience in studying the HH hierarchy, we can make, in the VZS limit, the following speculations about the spin states of (20):

(i) all even-numerator states [such as $\nu = 2/(2k+1)$] are spin unpolarized;

(ii) all even-denominator states are also spin unpolarized;

(iii) all states with both the numerator and denominator (of filling fraction) odd are partially/fully polarized.

We expect that the above speculations can be testified experimentally in the near future.

It should be noted that Lopez and Fradkin [9] have shown that the double-layer systems can be treated by the fermion Chern–Simons (CS) theory, namely, a theory of interacting fermions (which has $\theta = 0$, with θ the coefficient of the CS action; for the double-layer systems θ is a 2×2 matrix) is equivalent to a family of theories of fermions with θ being such that the electrons are attached to an even number of fluxes of the CS gauge field in their own layer and to an arbitrary number of fluxes of the CS gauge field in the opposite layer. In this form, the theory has a $U(1) \otimes U(1)$ gauge invariance. A direct inspection of the allowed fractions, (20) and (2.13) [9] shows that they do not yield the same allowed fractions. Therefore, the two hierarchies are not completely equivalent.

In the following section, we further generalize these considerations to triple- and multi-layer systems.

3. Hierarchy for triple- and multi-layer systems

In this section, we generalize the arguments in the previous section to triple- and multi-layer systems. First of all, let us write down the counterpart of (1) for triple-layer systems.

$$\begin{aligned}\psi^{(m_1,m_2,m_3,n_{12},n_{23},n_{13})}[u_i^1, v_i^1, u_i^2, v_i^2, u_i^3, v_i^3] &= \prod_{i<j}^{N_1}(u_i^1 v_j^1 - u_j^i v_i^1)^{m_1} \prod_{i<1}^{N_2}(u_i^2 v_j^2 - u_j^2 v_i^2)^{m_2} \\ &\times \prod_{i<j}^{N_3}(u_i^3 v_j^3 - u_j^3 v_i^3)^{m_3} \prod_{i,j}^{N_1 N_2}(u_i^1 v_j^2 - u_j^2 v_i^1)^{n_{12}} \prod_{i,j}^{N_2 N_3}(u_i^2 v_j^3 - u_j^3 v_i^2)^{n_{23}} \\ &\times \prod_{i,j}^{N_1 N_3}(u_i^1 v_j^3 - u_j^3 v_i^1)^{n_{13}} \end{aligned} \tag{21}$$

where u_i^1, v_i^1, u_i^2, v_i^2 and u_i^3, v_i^3 are the spinor coordinates for the electrons in the first, second and third layers, respectively. Similar to (8) and (9), we have the quasihole and quasielectron wavefunctions for triple-layer systems as follows:

$$\begin{aligned}\psi_{QH}^{(m_1,m_2,m_3,n_{12},n_{23},n_{13})}[u_i^1, v_i^1, u_i^2, v_i^2, u_i^3, v_i^3] &= (21) \times \prod_{i=1}^{N_{ex,1}}(\beta_1 u_i^1 - \alpha_1 v_i^1) \\ &\times \prod_{i=1}^{N_{ex,2}}(\beta_2 u_i^2 - \alpha_2 v_i^2) \prod_{i=1}^{N_{ex,3}}(\beta_3 u_i^3 - \alpha_3 v_i^3) \end{aligned} \tag{22}$$

$$\begin{aligned}\psi_{QE}^{(m_1,m_2,m_3,n_{12},n_{23},n_{13})}[u_i^1, v_i^1, u_i^2, v_i^2, u_i^3, v_i^3] &= (21) \times \prod_{i=1}^{N_{ex,1}}[\beta_1^*(\partial/\partial u_i^1) - \alpha_1^*(\partial/\partial v_i^1)] \\ &\times \prod_{i=1}^{N_{ex,2}}[\beta_2^*(\partial/\partial u_i^2) - \alpha_2^*(\partial/\partial v_i^2)] \prod_{i=1}^{N_{ex,3}}[\beta_3^*(\partial/\partial u_i^3) - \alpha_3^*(\partial/\partial v_i^3)]. \end{aligned} \tag{23}$$

Using the operator (2) with $k = 1, 2, 3$ with the wavefunctions (22) and (23), we find

$$\begin{pmatrix} 2S_1 \\ 2S_2 \\ 2S_3 \end{pmatrix} = \begin{pmatrix} m_1 & n_{12} & n_{13} \\ n_{12} & m_2 & n_{23} \\ n_{13} & n_{23} & m_3 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} - \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} + \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \begin{pmatrix} N_{ex,1} \\ N_{ex,2} \\ N_{ex,3} \end{pmatrix} \tag{24}$$

where m_1, m_2 and m_3 are odd, n_{12}, n_{23} and n_{13} even and $\alpha_{1,2,3} = 1$ for quasiholes and $\alpha_{1,2,3} = -1$ for quasielectrons. Symbolically, (24) can be expressed by (11), but in this case $[m]$ is a 3×3 matrix and $\{m\}$ a three-dimensional vector.

Similarly, the form of (18) is applicable to the triple-layer systems with

$$[m] = \begin{pmatrix} m_1 & n_{12} & n_{13} \\ n_{12} & m_2 & n_{23} \\ n_{13} & n_{23} & m_3 \end{pmatrix} \qquad [p_i] = \begin{pmatrix} p_{i1} & q_{i12} & q_{i13} \\ q_{i12} & p_{i2} & q_{i23} \\ q_{i13} & q_{i23} & p_{i3} \end{pmatrix}$$
$$\alpha_i = \begin{pmatrix} \alpha_{i1} & 0 & 0 \\ 0 & \alpha_{i2} & 0 \\ 0 & 0 & \alpha_{i3} \end{pmatrix} \qquad i = 1, 2, \ldots, n. \tag{25}$$

Let us discuss two special cases of (25).

(i) For

$$\begin{aligned} &m_1 = m_2 = m_3 = m \qquad n_{12} = n_{23} = n_{13} = n \\ &p_{i1} = p_{i2} = p_{i3} = p_i \qquad q_{i12} = q_{i23} = q_{i13} = q_i \\ &\alpha_{i1} = \alpha_{i2} = \alpha_{i3} = \alpha_i = \pm 1 \end{aligned} \tag{26}$$

(18) is reduced to

$$\nu = \frac{1}{m + 2n + \alpha_1/\{p_1 + 2q_1 + \alpha_2/[p_2 + 2q_2 + \cdots + \alpha_n/(p_n + 2q_n)]\}}. \tag{27}$$

Physically, (26) implies that

(a) for a primary state (m, m, m, n, n, n), the intralayer correlations in three layers are equal to m and the interlayer correlations between any two layers are n;

(b) all daughter states $(p_i, p_i, p_i, q_i, q_i, q_i)$ have similar correlations as the parent state (m, m, m, n, n, n), namely, at the ith level of the hierarchy, the intralayer correlations in three layers are equal to p_i and the interlayer correlations between any two layers are q_i;

(c) $\alpha_i = 1$ or -1 denotes that the constituents of the ith-level daughter state are all charged according to the same sign, implying repulsive correlations between quasiparticles in both intralayer and any two layers.

(ii) Let us then consider a primary state (m_1, m_2, m_3, n, n) characterized by

$$\begin{pmatrix} m_1 & n & 0 \\ n & m_2 & n \\ 0 & n & m_3 \end{pmatrix}. \tag{28}$$

Physically, (28) implies that the intralayer correlations in the ith layer are characterized by m_i $(i = 1, 2, 3)$ and each layer correlates only with its nearest layers as well as all interlayer correlations are equal to n. Calculating the sum of all the entries of the ith column (or row) of the inverse matrix of (28), we find that the filling fractions for each layer are

$$\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} m_3(m_2 - n) \\ m_1 m_3 - (m_1 + m_3) \cdot n \\ m_1(m_2 - n) \end{pmatrix} \tag{29}$$

where $\Delta = m_1 m_2 m_3 - (m_1 + m_3) \cdot n^2$. Some examples of (29) have been discussed by MacDonald in [10]. Especially, when $m_i = 3$ and $n = 1$, $\nu = 5/7$. This should be the triple-layer system analogue of the $\nu = 1/2$ effect observed by Eisenstein *et al* [6] and others [7].

On the basis of the discussions mentioned above, (18) is also applicable to multi-layer systems. For M-layer systems, (25) becomes

$$[m]_M = \begin{pmatrix} m_1 & n_{12} & \dots & n_{1M} \\ n_{12} & m_2 & \dots & \vdots \\ \vdots & & \ddots & \vdots \\ n_{1M} & \dots & \dots & m_M \end{pmatrix} \qquad [p_i]_M = \begin{pmatrix} p_{i1} & q_{i12} & \dots & q_{i1M} \\ q_{i12} & p_{i2} & \dots & \vdots \\ \vdots & & \ddots & \vdots \\ q_{i1M} & \dots & \dots & p_{iM} \end{pmatrix}$$

$$\alpha_i = \begin{pmatrix} \alpha_{i1} & 0 & \dots & 0 \\ 0 & \alpha_{i2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{iM} \end{pmatrix} \qquad i = 1, 2, \dots, n. \tag{30}$$

Similarly, we consider two special cases of (30).

(i) For all diagonal elements of the matrices $[m]_M$ and $[p_i]_M$ are equal to m and p_i, respectively, and all off-diagonal elements are n and q_i, respectively, and $\alpha_{ij} = \pm 1$, $j = 1, 2, \dots, M$, (18) is reduced to

$$\nu = \cfrac{1}{m + (M-1)n + \cfrac{\alpha_1}{p_1 + (M-1)q_1 + \cfrac{\alpha_2}{p_2 + (M-1)q_2 + \ddots + \cfrac{\alpha_n}{p_n + (M-1)q_n}}}}. \tag{31}$$

Following the discussions mentioned above, the physical implications leading to (31) are clear. Obviously, (20) and (27) are the special cases of (31) for $M = 2$ and 3, respectively.

(ii) Let us consider the following symmetric tridiagonal coupling matrix:

$$\begin{pmatrix} m_1 & n & 0 & \dots & 0 & 0 \\ n & m_2 & n & \dots & 0 & 0 \\ 0 & n & m_3 & & & \vdots \\ \dots & \dots & & \ddots & & \vdots \\ \dots & \dots & & & \ddots & n \\ 0 & 0 & \dots & \dots & n & m_M \end{pmatrix}. \tag{32}$$

This matrix is very interesting because it corresponds to the nearest-neighbour interlayer correlation which is more likely to be realized experimentally, Especially, when the number of layers M is large, the electron system may be regarded as three dimensional [11]. In fact, if one has the technology to grow many layers of heterojunctions, it is even possible to realize a three-dimensional FQHE in the laboratory described by the coupling matrix (32).

4. Conclusions

The hierarchy of quantized Hall states in the double-layer electron systems has been straightforwardly extended to the cases of triple-layer as well as multi-layer electron systems by developing a matrix continued fraction form for the filling factors. We have shown that under some physical restrictions the matrix continued fraction form can be expressed as the usual numerical continued fraction (31), which contains most of the filling fractions appearing in the literature. The most manifest property of our formulation (31) is that each layer has the same filling fraction with either odd or even denominator according to the

concrete values of the parameters in the matrices (30). We believe that with the progress of the technology, many fractional quantum states included in (31) will be discovered experimentally in the near future.

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References

[1] Haldane F D M 1983 *Phys. Rev. Lett.* **51** 605
Haldane F D M 1987 *The Quantum Hall Effect* ed R E Prange and S M Girvin (New York: Springer) p 175
[2] Halperin B I 1984 *Phys. Rev. Lett.* **52** 1583
[3] Greiter M and McDonald I A 1993 *Nucl. Phys.* B **410** 521
[4] Chakraborty T and Pietilainen P 1990 *Phys. Rev.* B **41** 10 862
[5] Engel I W, Hwang S W, Sajoto T, Tsui D C and Shayegan M 1992 *Phys. Rev.* B **45** 3418
[6] Eisenstein J P, Boebinger G S, Pfieffer L N, West K W and He S 1992 *Phys. Rev. Lett.* **68** 1383
[7] Suen Y W, Engel L W, Santos M B, Shayegan M and Tsui D C 1992 *Phys. Rev. Lett.* **68** 1379
[8] See, for example, Cappelli A, Trugenberger C A and Zemba G R 1995 *Nucl. Phys.* B **448** 470
[9] Lopez A and Fradkin E 1995 *Phys. Rev.* B **51** 4347
[10] MacDonald A H 1990 *Surf. Sci.* **229** 1
[11] This kind of 2D electron gas multilayer sample does exist. It is the topic of current investigation. See, for example, Balents L and Fisher M P A 1996 *Phys. Rev. Lett.* **76** 2782 and references therein.